Order $1/N^2$ test of the Maldacena conjecture: Cancellation of the one-loop Weyl anomaly.

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Abstract

We test the Maldacena conjecture for type IIB String Theory/ $\mathcal{N} = 4$ Yang-Mills by calculating the one-loop corrections in the bulk theory to the Weyl anomaly of the boundary CFT when the latter is coupled to a Ricci flat metric. The contributions cancel within each supermultiplet, in agreement with the conjecture.

The truly remarkable nature of the AdS/CFT correspondence is nicely illustrated by Henningson and Skenderis' beautiful computation (from the bulk) of the leading order contribution to the boundary theory Weyl anomaly [1]. This test of Maldacena's conjecture [3] demonstrates that, at large N, an (exact) one-loop calculation in four-dimensional $\mathcal{N}=4$ Super-Yang-Mills theory can be reproduced by a tree-level calculation in General Relativity.

When the boundary Yang-Mills theory is coupled to a non-dynamical, external metric, g_{ij}, the Weyl anomaly, $\mathcal{A}$, is the response of the free energy, $F[g_{ij}]$, to a scale transformation of that metric:

$$\delta F = \int d^4x\, \sqrt{g}\delta\rho\mathcal{A}, \qquad \delta g_{ij} = \delta\rho g_{ij}. \tag{1}$$

On general grounds $\mathcal{A}$ must be a linear combination of the Euler density, E, and the square of the Weyl tensor, I. A well-known one-loop calculation gives

$$\pi^2\mathcal{A} = -(N^2-1)(E+I), \tag{2}$$

and supersymmetry protects this from higher-loop corrections. The tree-level calculation in the bulk reproduces the leading N^2 piece, by solving the Einstein equations perturbatively near the boundary. We would expect that the -1 piece is due to loops in the bulk, but these depend on much more than just classical General Relativity, and reproducing them provides a more stringent test of the Maldacena conjecture sensitive to the detailed particle content of the bulk theory.

In this letter we describe the results of the one-loop calculation in the bulk when the boundary metric is Ricci flat. This restriction enables us to avoid having to solve for the metric perturbatively; the bulk metric

$$ds^2 = \frac{1}{t^2}\left(l^2\, dt^2 + \sum_{i,j} g_{ij}\, dx^i\, dx^j\right), \qquad t>0 \tag{3}$$

satisfies the Einstein equations with cosmological constant $\Lambda = -6/l^2$. For a Ricci-flat metric $E = -I = R^{ijkl}R_{ijkl}/64$, so that (2) vanishes. Our computation gives the coefficient of the square of the Riemann tensor, which is one half of the information needed to fix the Weyl anomaly completely. For the Maldacena conjecture to hold this must be zero. We work in the field theory limit of String Theory, but we will argue that corrections from higher string modes vanish.

The spectrum of the compactification on $AdS_5 \times S^5$ of Type IIB Supergravity was obtained in [4] by perturbing classical equations of motion about a solution in which all the fields vanish except the metric and a self-dual five-form field strength. These classical equations do not follow from an action principle, but rather are chosen to be compatible with supersymmetry. (If there had been an action this calculation would have given its second functional derivative.)

Since we work with a more general background than the maximally symmetric AdS_5 we need to repeat the calculation. We have done this, and also constructed a corresponding Lagrangian which we need in order to go beyond the classical level and quantize the theory. The main complication is that some of the equations of motion used in [4] are no longer present when we impose all the gauge conditions, as we must in order to do one-loop calculations. We constructed our Lagrangian *after* expanding in S^5 spherical harmonics, so it is local in AdS_5, but *not* in the full ten-dimensional space. In this way we avoid the paradox associated with a Lagrangian description of self-dual field strengths.

The conclusion of our analysis, perhaps unsurprisingly, is that the spectrum of physical particles is the same as in [4], so we leave a detailed description of the calculation to another place.

The central object of interest in the AdS/CFT correspondence is the 'partition function' given as a functional integral for the bulk theory in which the fields have precribed values, φ, on the 'boundary' at $t = 0$ [2]. Because of the divergences in the metric it is necessary to introduce a small t cut-off, τ', even in tree-level calculations. At one-loop it is also necessary to introduce a large t cut-off τ; this introduces another boundary, and the functional integral should be performed with the fields taking prescribed values, $\tilde{\varphi}$, there as well. Consequently the partition function is the limit as the cut-offs are removed of a functional $\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi]$. The exponential of the free energy is the field independent part of this partition function. With the regulators in place the free energy becomes a function of τ,τ', and the Weyl anomaly can be found by exploiting the invariance of the five-dimensional metric (3) under $t \to (1+\delta\rho/2)t$, $g_{ij} \to (1+\delta\rho)g_{ij}$, with $\delta\rho$ constant. So, for a constant Weyl scaling

$$\delta F = \int d^4x\,\sqrt{g}\delta\rho\mathcal{A} = -\frac{\delta\rho}{2}\left(\tau\frac{\partial F}{\partial\tau} + \tau'\frac{\partial F}{\partial\tau'}\right) \tag{4}$$

At one-loop we only need the quadratic fluctuations in the action, so the fields are essentially free. In [5] we computed the Weyl anomaly for free scalar and spin-half particles for the metric (3), not by performing a functional integration but by interpreting $\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi]$ (after Wick rotation of g_{ij}) as the Schrödinger functional, i.e. the matrix element of the time evolution operator between eigenstates of the field,

$$\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi] = \langle\,\tilde{\varphi}\,|\,T\,\exp(-\int_{\tau'}^{\tau} dt\,H(t))\,|\,\varphi\,\rangle\,, \tag{5}$$

To illustrate this consider a massless scalar field. Ψ satisfies the functional Schrödinger equation

$$\frac{\partial}{\partial\tau}\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi] = \frac{1}{2}\int d\mathbf{x}\,\left(\tau^3\frac{\delta^2}{\delta\tilde{\varphi}^2} + \Omega\,\tilde{\varphi}\nabla\cdot\nabla\tilde{\varphi} + 2\,\delta^4(0)/\tau\right)\Psi_{\tau,\tau'}[\tilde{\varphi},\varphi], \tag{6}$$

with a similar equation for the τ' dependence, and the initial condition that Ψ becomes a delta-functional as τ' approaches τ. The logarithm of Ψ takes the form

$$F + \int d^d\mathbf{x}\,\left(\frac{1}{2}\tilde{\varphi}\,\Gamma_{\tau,\tau'}\,\tilde{\varphi} + \tilde{\varphi}\,\Xi_{\tau,\tau'}\,\varphi + \frac{1}{2}\varphi\,\Upsilon_{\tau,\tau'}\,\varphi\right), \tag{7}$$

so that the Schrödinger equations relate the derivatives of F with respect to τ and τ' that appear in (4) to the functional traces of $\Gamma_{\tau,\tau'}$ and $\Upsilon_{\tau,\tau'}$. These operators are readily obtained by solving the Schrödinger equations in powers of $\nabla\cdot\nabla$ so that eventually we obtain $\mathcal{A}$ in terms of the heat-kernel associated with this four-dimensional operator. We found that when a mass is included $\mathcal{A}$ vanishes for generic values of the mass. However, it is not a continuous function, and when the mass is such that $\sqrt{l^2m^2+4}$ is an integer, N, then $\mathcal{A} = -N\, R^{ijkl}R_{ijkl}/(5760\pi^2)$. Similarly, for spin-half fermions, $\mathcal{A}$ is non-vanishing when $|lm| + 1/2$ is an integer. These are precisely the mass values that occur in the spectrum of Supergravity.

Although our original calculation only applied to scalar and spin-half fields, it gives us most of the information we need to calculate $\mathcal{A}$ for fields of *arbitrary* spin. The strategy is to decompose the fields in such a way that they satisfy functional equations of a similar form as those satisfied by their lower spin counterparts. For example, consider a massive vector field A_μ. The Lagrangian density, as usual, is proportional to the square of the field strength. Thus A_0 is non-dynamical, and may be integrated out of the partition function. Now the *transverse* part of A_i has a Lagrangian which is identical in form to the scalar field Lagrangian, and the calculation we outlined above holds for this field with only two differences: the heat-kernel of $\nabla \cdot \nabla$ is the one acting on transverse vector fields, and the inner-product on A_i has a measure which differs by a factor of t^2 from the inner-product on scalars; this has the effect that the mass values for which the anomaly is non-zero are changed to $\sqrt{l^2m^2+1} = N$.

So what about the remaining longitudinal mode? The trick is to make a change of variables so that the Jacobian cancels the unwanted determinant coming from integrating out A_0. Now the resulting Lagrangian is not quite the usual Lagrangian for scalar fields, but it is the same to lowest order in $\nabla \cdot \nabla$, which is all we need to calculate the anomaly. The mass term is such that the anomaly is again non-zero for $\sqrt{l^2m^2+1} = N$.

Finally, the heat-kernels of $\nabla \cdot \nabla$ acting on a transverse vector and a scalar sum to the heat-kernel of $\nabla \cdot \nabla$ acting on an *unconstrained* vector. So the anomaly arising from a vector field in *five* dimensional AdS spacetime is obtained from the heat kernel of the Laplacian acting on *four* dimensional vector fields. Furthermore, the anomaly is non-zero for integer values of $\sqrt{l^2m^2+1}$, which are again precisely the mass values appearing in the Supergravity spectrum.

The same thing happens for all the fields of the theory. For generic values of the masses the Weyl anomalies vanish, but for those values that occur in the spectrum $\mathcal{A}$ is non-zero. In general, the anomaly is given by

$$\mathcal{A} = -\alpha\, N\, R^{ijkl}R_{ijkl}/(5760\pi^2), \tag{8}$$

where α is the coefficient coming from the relevant heat kernel, normalized to 1 for scalar fields. We list the values of N and α for all the massive fields in table 1. So, although our test of the Maldacena conjecture could have been passed trivially by all the one-loop contributions vanishing separately, we have instead non-vanishing contributions from an infinite number of fields corresponding to the Kaluza-Klein modes on S^5.

Now the mass spectrum is given in table 2, where the supermultiplets are labelled by an integer p. The first three fields in the table form the doubleton representation for $p = 1$, but this is not present in the spectrum. The $p = 2$ or massless multiplet contains all the gauge fields, which we will discuss in a moment. For the $p \geq 3$ multiplets the total anomaly is proportional to $\sum_{\text{fields}} \alpha N r$ where r is the dimension of the $SU(4) \sim SO(6)$ representation in which the fields transform. It can be seen from the table that r is always

Table 1: Anomaly coefficients of massive fields on AdS_5.

Field	N	α
ϕ	$\sqrt{m^2+4}$	1
ψ	$\|m\|+1/2$	7/2
A_μ	$\sqrt{m^2+1}$	-11
$A_{\mu\nu}$	m	33
ψ_μ	$\|m\|+1/2$	-219/2
$h_{\mu\nu}$	$\sqrt{m^2+4}$	189

a polynomial of degree 4 in p, while N is linear in p. So each field contributes a polynomial of degree 5 to the sum.

There is one field for which the situation is not quite as simple as we have made out. For the gravitino the extra factor of t^2 in the inner-product changes the coefficient in front of the anomaly from N to $N-1/2$. For ψ_0 it is changed from N to $N-1$. All this means that the gravitino contributes $(\alpha N - 46)r$ instead of $\alpha N r$ to the above sum. Once we have taken this into consideration we find that the sum, and thus the anomaly, vanishes for each of the $p \geq 3$ multiplets.

Before we discuss the massless multiplet, notice that the result so far is in accordance with the existence of a consistent truncation which essentially discards all of the $p \geq 3$ multiplets. Even if the overall anomaly was non-zero, the contribution from these multiplets should still vanish.

So we turn to the $SO(6)$ gauge fields of the massless multiplet. The story is still much the same as before, but we need to fix the gauge symmetries and include Faddeev-Popov ghosts. In table 3 we show how the decomposition works for each of the gauge fields. Using this information it is easy to see that the anomaly sums to zero for this multiplet also, so that the Maldacena conjecture has successfully passed our test.

To summarize. The coefficient of the square of the Riemann tensor in the Weyl anomaly of $\mathcal{N}=4$ Super Yang-Mills theory is zero to all loops. For the Maldacena conjecture to hold the same must be true in the calculation on the AdS_5 side. We found that at one-loop the contributions of the individual species of fields was non-zero, however they sum to zero over each supermultiplet, in agreement with the conjecture. There is a similar cancellation over supermultiplets in the case of the chiral anomaly [6].

Our calculation was done in the field theory limit of String Theory, however we expect that the neglected string modes have contributions that vanish separately. This is because the anomaly is not a continuous function of mass, but vanishes for generic values. It is non-zero for the Kaluza-Klein spectrum of supergravity which has a special property. A standard construction of Anti-de-Sitter space is as a hyperboloid in $R^{2,4}$. This contains closed time-like curves which are removed by going to the universal cover. Curiously the spectrum of supergravity allows the corresponding fields to be single-valued on this hyperboloid. This property will not be shared by the higher string modes whose masses depend on the string-scale α'.

Table 2: Mass spectrum. The supermultiplets (irreps of U(2,2/4)) are labelled by the integer p. Note that the doubleton ($p=1$) does not appear in the spectrum. The (a,b,c) representation of $SU(4)$ has dimension $r=(a+1)(b+1)(c+1)(a+b+2)(b+c+2)(a+b+c+3)/12$, and a subscript c indicates that the representation is complex. (Spinors are four component Dirac spinors in AdS_5).

Field	$SO(4)$ rep$^{\rm n}$	$SU(4)$ rep$^{\rm n}$	Mass on S^5
$\phi^{(1)}$	$(0,0)$	$(0,p,0)$	$m^2=p(p-4), \quad p\geq 2$
$\psi^{(1)}$	$(\frac{1}{2},0)$	$(0,p-1,1)_c$	$m=p-3/2, \quad p\geq 2$
$A^{(1)}_{\mu\nu}$	$(1,0)$	$(0,p-1,0)_c$	$m^2=(p-1)^2, \quad p\geq 2$
$\phi^{(2)}$	$(0,0)$	$(0,p-2,2)_c$	$m^2=(p+1)(p-3), \quad p\geq 2$
$\phi^{(3)}$	$(0,0)$	$(0,p-2,0)_c$	$m^2=(p+2)(p-2), \quad p\geq 2$
$\psi^{(2)}$	$(0,0)$	$(0,p-2,1)_c$	$m=p-1/2, \quad p\geq 2$
$A^{(1)}_{\mu}$	$(\frac{1}{2},\frac{1}{2})$	$(1,p-2,1)$	$m^2=p(p-2), \quad p\geq 2$
$\psi^{(1)}_{\mu}$	$(1,\frac{1}{2})$	$(1,p-2,0)_c$	$m=p-1/2, \quad p\geq 2$
$h_{\mu\nu}$	$(1,1)$	$(0,p-2,0)$	$m^2=(p+2)(p-2), \quad p\geq 2$
$\psi^{(3)}$	$(\frac{1}{2},0)$	$(2,p-3,1)_c$	$m=p-1/2, \quad p\geq 3$
$\psi^{(4)}$	$(\frac{1}{2},0)$	$(0,p-3,1)_c$	$m=p+1/2, \quad p\geq 3$
$A^{(2)}_{\mu}$	$(\frac{1}{2},\frac{1}{2})$	$(1,p-3,1)_c$	$m^2=(p+1)(p-1), \quad p\geq 3$
$A^{(2)}_{\mu\nu}$	$(1,0)$	$(2,p-3,0)_c$	$m^2=p^2, \quad p\geq 3$
$A^{(3)}_{\mu\nu}$	$(1,0)$	$(1,p-3,0)_c$	$m^2=(p+1)^2, \quad p\geq 3$
$\psi^{(2)}_{\mu}$	$(1,\frac{1}{2})$	$(1,p-3,0)_c$	$m=p+1/2, \quad p\geq 3$
$\phi^{(4)}$	$(0,0)$	$(2,p-4,2)$	$m^2=(p+2)(p-2), \quad p\geq 4$
$\phi^{(5)}$	$(0,0)$	$(0,p-4,2)_c$	$m^2=(p+3)(p-1), \quad p\geq 4$
$\phi^{(6)}$	$(0,0)$	$(2,p-4,2)$	$m^2=p(p+4), \quad p\geq 4$
$\psi^{(5)}$	$(\frac{1}{2},0)$	$(2,p-4,1)_c$	$m=p+1/2, \quad p\geq 4$
$\psi^{(6)}$	$(\frac{1}{2},0)$	$(0,p-4,1)_c$	$m=p+3/2, \quad p\geq 4$
$A^{(3)}_{\mu}$	$(\frac{1}{2},\frac{1}{2})$	$(1,p-4,1)$	$m^2=(p+3)(p-1), \quad p\geq 4$

Table 3: Decomposition of gauge fields for the massless multiplet. The spin-connection leads to a correction $N \rightarrow N - 1/2$ for ψ_i and $N \rightarrow N - 1$ for ψ_0. Some components of the graviton have non-physical mass values, but cancel against identical Fadeev-Popov ghosts.

Original field	Gauge fixed fields	N	α
A_μ	A_i	1	-11
(**15** of $SU(4)$)	A_0	2	1
	b_{FP}, c_{FP}	2	-1
ψ_μ	ψ_i	2	-106
(**4** of $SU(4)$)	ψ_0	2	7/2
	λ_{FP}, ρ_{FP}	3	-7/2
	σ_{GF}	2	-7/2
$h_{\mu\nu}$	$h_{ij}^{traceless}$	2	189
($SU(4)$ singlet)	h_{0i}	3	-11
	h_{00}, h^μ_μ	cancel	B_0^{FP},C_0^{FP}
	B_0^{FP},C_0^{FP}	cancel	h_{00}, h^μ_μ
	B_i^{FP},C_i^{FP}	3	11

We have calculated one half of the Weyl anomaly. The remaining terms that depend on the Ricci tensor can be computed by a similar calculation but in a different background, say one corresponding to a spherical boundary, for which the Einstein equations can again be solved exactly. This would provide a further test, as would the computation of higher loops in the bulk.

An essential part of our calculation was the use of the functional Schrödinger equation which we consider to be a useful tool in the study of the AdS/CFT correspondence, not least because the central object of study in the correspondence is itself the Schrödinger functional.

References

[1] M. Henningson and K. Skenderis JHEP 9807 (1998), 023

[2] E. Witten, Adv.Theor.Math.Phys.2 (1998), 253.

[3] J. Maldacena, Adv.Theor.Math.Phys.2 (1998), 231.

[4] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, Phys. Rev. D32 (1985), 389.

[5] P. Mansfield and D. Nolland, JHEP 9907 (1999), 028.

[6] A. Bilal and C-S. Chu, Nucl.Phys. B562 (1999), 181.